Milton Math Tournament
Varsity Team Ciphering

1. Solve the following inequality on $(0, 2\pi)$

$$\sin x < \cos x < \tan x < \cot x$$

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2. Let $f(x)$ be a twice-differentiable function such that $f(x)$ and $f'(x)$ are one-to-one and

$$(f(x))^2 \cdot f''(f^{-1}(x)) = -f'(x) \cdot f'(f^{-1}(x)) \cdot f''\left(\frac{1}{f(x)}\right)$$

If $f(7) = 9$, find $f^{-1}(7) + f(9)$.

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3. Given that

$$\sum_{k=0}^{\infty} \frac{k^3}{11^k} = \frac{a}{b}$$

where a and b are relatively prime integers, find the value of $a + b$.

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4. The following system of linear congruences is solved for all $x = ak + b$, where k is any integer.

$$\begin{aligned} 1234x + 123 &\equiv 4321 \pmod{24} \\ 4321x + 4545 &\equiv 9337 \pmod{16} \end{aligned}$$

Find $a + b$ if b is as small as possible without being negative.

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5. A horrible disease has been going around in your town. The probability of having this disease is 1/2. The medical test for the disease, like all tests, isn't perfect, and so the probability of a positive result is 4/5 when you have the disease and 1/5 when you don't. Not only that, but the test is risky! Testing preliminaries require you to be slightly exposed to the disease, so that the nth time you have the test done, you run the risk of contracting the disease with a probability of $1/(n+1)$. After having the test done two times, and both times yielding positive, the probability that you have the disease is a/b, where a and b are relatively prime integers. Find $a + b$.

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6. A triangle has sides a, b, and c. Define the function $K(a, b, c)$ as

$$K(a, b, c) = \sqrt{F(a, b, c) - \frac{1}{16}(a^2 + b^2 - c^2)^2}$$

If $K(a, b, c)$ is the area of the triangle, find $F(a, b, c)$.

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7. Find the sum $x + y$ for (x, y) satisfying the system of equations

$$\frac{31}{2x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$

$$\frac{31}{2x} - \frac{1}{2y} = 2(y^4 - x^4)$$

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8. A triangle has sides of length 1, r, and r^2. If the area of the triangle has the same numerical value as the perimeter, then r satisfies

$$r^8 + 17r^4 + 32r^3 + Kr^2 + 32r + 17 = 2r^6$$

Find K.

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9. A well-known identity in combinatorics is as follows:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

What is the name of this identity (i.e., whose name is associated with it)?

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10. Let X be a random variable representing the number of hairs on a person's head in an infinite population. The mean of X is μ, and the standard deviation is σ. If the random variable Y represents the total number of hairs on two peoples heads (the two being chosen at random from the same population), what is the standard deviation of Y?